UNIFYING IMAGE PLANE LIFTINGS FOR CENTRAL CATADIOPTRIC AND DIOPTRIC CAMERAS

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Abstract In this paper, we study projection systems with a single viewpoint, including combinations of mirrors and lenses (catadioptric) as well as just lenses with or without radial distortion (dioptric systems). Firstly, we extend a well-known unifying model for catadioptric systems to incorporate a class of dioptric systems with radial distortion. Secondly, we provide a new representation for the image planes of central systems. This representation is the lifting through a Veronese map of the original image plane to the 5D projective space. We study how a collineation in the original image plane can be transferred to a collineation in the lifted space and we find that the locus of the lifted points which correspond to projections of world lines is a plane in parabolic catadioptric systems and a hyperplane in case of radial lens distortion.

Keywords: Central Catadioptric Cameras, Radial Distortion, Lifting of Coordinates, Veronese Maps

1. Introduction

A vision system has a single viewpoint if it measures the intensity of light traveling along rays which intersect in a single point in 3D (the projection center). Vision systems satisfying the single viewpoint constraint are called central projection systems. The perspective camera is an example of a central projection system. The mapping of points in the scene into points in the image is linear in homogeneous coordinates, and can be described by a 3×4 projection matrix $\mathbf{P}$ (pin-hole model). Perspective projection can be modeled by intersecting a plane with a pencil of lines going through the scene points and the projection center $\mathbf{O}$.

There are central projection systems whose geometry can not be described using the conventional pin-hole model. In [1] Baker et al. derive the entire class of catadioptric systems verifying the single viewpoint constraint. Sensors with a wide field of view and a unique projection center can be built by combining a hyperbolic mirror with a perspective camera, and a parabolic mirror with an orthographic camera (paracatadioptric system). However the mapping between points in the 3D world and points in the image is non-linear. In [15] it is shown that in general the central catadioptric projection of a line is a conic section. A unifying theory for central catadioptric systems has been proposed in [9]. It is proved that central catadioptric image formation is equivalent to a projective mapping from a sphere to a plane with a projection center on a sphere axis perpendicular to the plane. Perspective cameras with non-linear lens distortion are another example of central projection systems where the relation in homogeneous coordinates between scene points and image points is no longer linear. True lens distortion curves are typically very complex and higher-order models are introduced to approximate the distortion during calibration [6, 17]. However, simpler low-order models can be used for many computer vision applications where an accuracy in the order of a pixel is sufficient. In this paper the radial lens distortion is modeled after the division model proposed in [8]. The division model is not an approximation to the classical model in [6], but a different approximation to the true curve. In this paper, we present two main novel results:

1. The unifying model of central catadioptric systems proposed [9] can be extended to include radial distortions. It is proved, that the projection in perspective cameras with radial distortion is equivalent to a projective mapping from a paraboloid to a plane, orthogonal to the paraboloid's axis, and with projection center in the vertex of the paraboloid. It is also shown that, assuming the division model, the image of a line is in general a conic curve.

2. For both catadioptric and radially distorted dioptric systems, we establish a new representation through lifting of the image plane to a five-dimensional projective space. In this lifted space, a collineation in the original plane corresponds to a collineation of the lifted points. We know that world line project to conic sections whose representatives in the lifted space lie on a quadric. We prove that in the cases of parabolic catadioptric projection and radial lens distortion this quadric degenerates to a hyperplane.

2. A Unifying Model for Perspective Cameras, Central Catadioptric Systems, and Lenses with Radial Distortion

In [9], a unifying model for all central catadioptric systems is proposed where conventional perspective imaging appears as a particular case. This section reviews this image formation model as well as the result that in general the catadioptric image of a line is a conic section [15]. This framework can be easily extended to cameras with radial distortion where the division model [8] is used to describe the lens distortion.

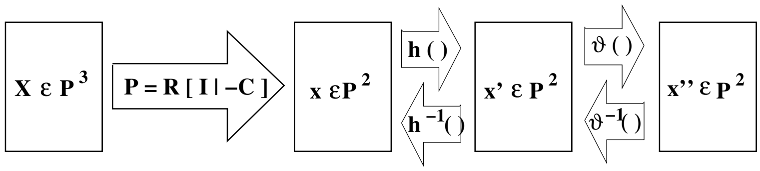

Figure 1. Steps of the unifying image formation model. The 3D point $\mathbf{X}$ is projected into point $\mathbf{x} = \mathbf{PX}$ assuming the conventional pin-hole model. To each point $\mathbf{x}$ corresponds an intermediate point $\mathbf{x}'$ which is mapped in the final image plane by function $\eth$. Depending on the sensor type, functions $\hbar$ and $\eth$ can represent a linear transformation or a non-linear mapping (see Tab 1).

This section shows that conventional perspective cameras, central catadioptric systems, and cameras with radial distortion underly one projection model. Fig. 1 is a scheme of the proposed unifying model for image formation. A point in the scene $\mathbf{X}$ is transformed into a point $\mathbf{x}$ by a conventional projection matrix $\mathbf{P}$. Vector $\mathbf{x}$ can be interpreted both as a 2D point expressed in homogeneous coordinates, and as a projective ray defined by points $\mathbf{X}$ and $\mathbf{O}$ (the projection center). Function $\hbar$ transforms $\mathbf{x}$ in the intermediate point $\mathbf{x}'$. Point $\mathbf{x}'$ is related with the final image point $\mathbf{x}''$ by function $\eth$. Both $\hbar$ and $\eth$ are transformations defined in the two dimensional oriented projective space. They can be linear or non-linear depending on the type of system, but they are always injective functions with an inverse. Tab. 1 summarizes the results derived along this section.

2.1 Perspective Camera and Central Catadioptric Systems

The image formation in central catadioptric systems can be split in three steps [4] as shown in Fig. 1: world points are mapped into an oriented projective plane by a conventional 3×4 projection matrix $\mathbf{P}$; the oriented projective plane is transformed by a non-linear function $\hbar$ (equation 1); the last step is a collineation in the plane $\mathbf{H_c}$ (equation 2). In this case, the function $\eth$ is a linear transformation depending on the camera intrinsics $\mathbf{K_c}$, the relative rotation between the camera and the mirror $\mathbf{R_c}$, and the shape of the reflective surface. As discussed in [9, 4], parameters ξ and ψ in equations 1 and 2, only depend on the system type and shape of the mirror. For paracatadioptric systems $\xi = 1$, while in the case of conventional perspective cameras $\xi = 0$. If the mirror is hyperbolic then ξ takes values in the range $]0, 1[$.

$$\mathbf{x}' = \hbar(\mathbf{x}) = (x, y, z + \xi\sqrt{x^2 + y^2 + z^2})^t \tag{1}$$

Perspective Camera ($\xi = 0, \psi = 0$)	
$\hbar(\mathbf{x}) = (x, y, z)^t$;	$\eth(\mathbf{x}') = \mathbf{K}\mathbf{x}'$
$\hbar^{-1}(\mathbf{x}') = (x', y', z')^t$;	$\eth^{-1}(\mathbf{x}'') = \mathbf{K}^{-1}\mathbf{x}''$
Hyperbolic Mirror ($0 < \xi < 1$)	
$\hbar(\mathbf{x}) = (x, y, z + \xi\sqrt{x^2 + y^2 + z^2})^t$;	$\eth(\mathbf{x}') = \mathbf{H_c}\mathbf{x}'$
$\hbar^{-1}(\mathbf{x}') = (x', y', z' - \frac{(x'^2+y'^2+z'^2)\xi}{z'\xi+\sqrt{z'^2+(1-\xi^2)(x'^2+y'^2)}})^t$;	$\eth^{-1}(\mathbf{x}'') = \mathbf{H_c}^{-1}\mathbf{x}''$
Parabolic Mirror ($\xi = 1$)	
$\hbar(\mathbf{x}) = (x, y, z + \sqrt{x^2 + y^2 + z^2})^t$;	$\eth(\mathbf{x}') = \mathbf{H_c}\mathbf{x}'$
$\hbar^{-1}(\mathbf{x}') = (2x'z', 2y'z', z'^2 - x'^2 - y'^2)^t$;	$\eth^{-1}(\mathbf{x}'') = \mathbf{H_c}^{-1}\mathbf{x}''$
Radial Distortion ($\xi < 0$)	
$\eth(\mathbf{x}') = (2x', 2y', z' + \sqrt{z'^2 - 4\xi(x'^2 + y'^2)})^t$,	$\hbar(\mathbf{x}) = \mathbf{K}\mathbf{x}$
$\eth^{-1}(\mathbf{x}'') = (x''z'', y''z'', z''^2 + \xi(x''^2 + y''^2))^t$,	$\hbar^{-1}(\mathbf{x}') = \mathbf{K}^{-1}\mathbf{x}'$;

Table 1. The mapping functions $\hbar$ and $\eth$ and corresponding inverses

$$\mathbf{x}'' = \underbrace{\mathbf{K}\mathbf{R_c}\begin{bmatrix} \psi - \xi & 0 & 0 \\ 0 & \xi - \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{H_c}} \hbar(\mathbf{x}) \tag{2}$$

The non-linear characteristics of the mapping are isolated in $\hbar$ which has a curious geometric interpretation. Since $\mathbf{x}'$ is a homogeneous vector representing a point in an oriented projective plane, $\lambda\mathbf{x}'$ represents the same point whenever $\lambda > 0$ [13]. Assuming $\lambda = 1/\sqrt{x^2 + y^2 + z^2}$ we obtain from equation 1 that

$$\begin{cases} x' = \frac{x}{\sqrt{x^2+y^2+z^2}} \\ y' = \frac{y}{\sqrt{x^2+y^2+z^2}} \\ z' - \xi = \frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

Assume $\mathbf{x}$ and $\mathbf{x}'$ as projective rays defined in two different coordinates systems in $\Re^3$. The origin of the first coordinate system is the effective viewpoint $\mathbf{O}$ and $\mathbf{x}$ is a projective ray going through $\mathbf{O}$. In a similar way $\mathbf{x}'$ represents a projective ray going through the origin $\mathbf{O}'$ of the second reference frame. According to the previous equation to each ray $\mathbf{x}$ corresponds one, and only one,

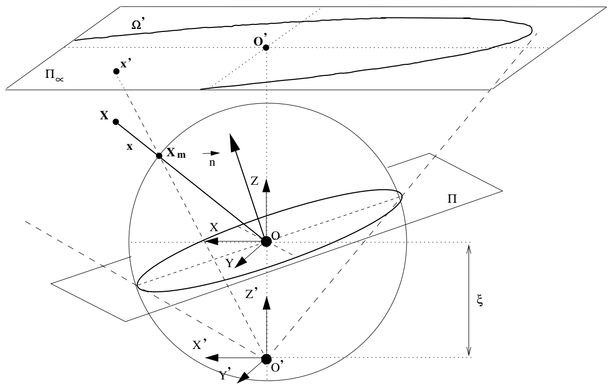

Figure 2. The sphere model for central catadioptric image formation. Projective ray $\mathbf{x}$ intersects the unitary sphere centered on the projection center $\mathbf{O}$ at point $\mathbf{X_m}$. The new projective point $\mathbf{x}'$ is defined by $\mathbf{O}'$ and $\mathbf{X_m}$. The distance between the origins $\mathbf{O}$ and $\mathbf{O}'$ is ξ which depends on the mirror shape

projective ray $\mathbf{x}'$. The correspondence is such that a pencil of projective rays $\mathbf{x}$ intersects a pencil of rays $\mathbf{x}'$ in a unit sphere centered in $\mathbf{O}$. The equation of the sphere in the coordinate system with origin in $\mathbf{O}'$ is

$$x'^2 + y'^2 + (z' - \xi)^2 = 1$$

We have just derived the well known sphere model derived in [9] and shown in Fig. 2. The homogeneous vector $\mathbf{x}$ can be interpreted as a projective ray joining a 3D point in the scene with the effective projection center $\mathbf{O}$, which intersects the unit sphere in a single point $\mathbf{X_m}$. Consider a point $\mathbf{O}'$ in $\Re^3$, with coordinates $(X, Y, Z) = (0, 0, -\xi)^t$ $(\xi \in [0, 1])$. To each $\mathbf{x}$ corresponds an oriented projective ray $\mathbf{x}'$ joining $\mathbf{O}'$ with the intersection point $\mathbf{X_m}$ in the sphere surface. The non-linear mapping $\hbar$ corresponds to projecting the scene in the unit sphere and then re-projecting the points on the sphere into a plane from a novel projection center $\mathbf{O}'$. Points in the image plane $\mathbf{x}''$ are obtained after a collineation $\mathbf{H_c}$ of the 2D projective points $\mathbf{x}'$ (equation 2).

Consider a line in space lying on a plane $\mathbf{\Pi}$ with normal $\mathbf{n} = (n_x, n_y, n_z)^t$, which contains the effective viewpoint $\mathbf{O}$ (Fig. 2). The 3D line is projected into a great circle on the sphere surface. The great circle is obtained by intersecting plane $\mathbf{\Pi}$ with the unit sphere. The projective rays $\mathbf{x}'$, joining $\mathbf{O}'$ with points in the great circle, form a central cone. The central cone, with vertex in $\mathbf{O}'$, projects into the conic $\mathbf{\Omega}'$ in the canonical image plane. The equation of $\mathbf{\Omega}'$ is provided in 3 and depends both on the normal $\mathbf{n}$ and on the parameter ξ [9, 4]. The original 3D line is projected in the catadioptric image on a conic section $\mathbf{\Omega}''$, which is the projective transformation of $\mathbf{\Omega}'$ ($\mathbf{\Omega}'' = \mathbf{H_c}^{-t}\mathbf{\Omega}'\mathbf{H_c}^{-1}$) [9, 15].

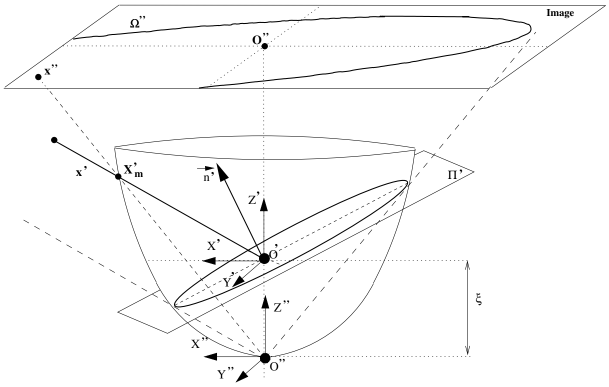

Figure 3. The paraboloid model for image formation in perspective cameras with lens with radial distortion. The division model for lens distortion is isomorphic to a projective mapping from a paraboloid to a plane with projection center on the vertex $\mathbf{O}''$. The distance between $\mathbf{O}''$ and the effective viewpoint is defined by the distortion parameter ξ

$$\mathbf{\Omega}' = \begin{bmatrix} n_x^2(1-\xi^2) - n_z^2\xi^2 & n_x n_y(1-\xi^2) & n_x n_z \\ n_x n_y(1-\xi^2) & n_y^2(1-\xi^2) - n_z^2\xi^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \tag{3}$$

Notice that the re-projection center $\mathbf{O}'$ depends only on mirror shape. For the case of a parabolic mirror $\mathbf{O}'$ lies in the sphere surface and the re-projection is a stereographic projection. For hyperbolic systems $\xi \in (0, 1)$ and point $\mathbf{O}'$ is inside the sphere in the negative Z-axis. The conventional perspective camera is a degenerate case of central catadioptric projection where $\xi = 0$ and $\mathbf{O}'$ is coincident with $\mathbf{O}$.

2.2 Dioptric Systems with Radial Distortion

In perspective cameras with lens distortion the mapping between points in the scene and points in the world can no longer be described in a linear way. In this paper the radial distortion is modeled using the so called division model [8]. According to the well known pin-hole model, to each point in the scene $\mathbf{X}$ corresponds a projective ray $\mathbf{x} = \mathbf{PX}$ which is transformed into a 2D projective point $\mathbf{x}' = \mathbf{Kx}$. Point $\mathbf{X}$ is projected in the image on point $\mathbf{x}''$, which is related with $\mathbf{x}'$ by a non-linear transformation that models the radial distortion. This transformation, originally introduced in [8], is provided in equation 4 where parameter ξ quantifies the amount of radial distortion. If $\xi = 0$ then points $\mathbf{x}'$ and $\mathbf{x}''$ are the same, and the camera is modeled as a conventional pin-hole. Equation 4 corresponds to the inverse of function $\eth$ (see Fig. 1 and Tab. 1), which isolates the non-linear characteristics of the mapping. In the case of dioptric systems with radial distortion function $\hbar$ is a linear transformation $\mathbf{K}$ (matrix of intrinsic parameters). Notice that the model of equation 4 requires that points $\mathbf{x}'$ and $\mathbf{x}''$ are referenced in a coordinate system with origin in the image distortion center. If the distortion center is not known in advance, we can place it at the image center without significantly affect the correction [17].

$$\mathbf{x}' = \eth^{-1}(\mathbf{x}'') = (x''z'', y''z'', z''^2 + \xi(x''^2 + y''^2))^t \tag{4}$$

Transformation $\eth$ has a geometric interpretation similar to the sphere model derived for central catadioptric image formation. As stated $\mathbf{x}'$ and $\lambda\mathbf{x}'$ represent the same point whenever λ is a positive scalar [13]. Assuming $\lambda = 1/\sqrt{x''^2 + y''^2}$ in equation 4 yields

$$\begin{cases} x' = \frac{x''z''}{x''^2+y''^2} \\ y' = \frac{y''z''}{x''^2+y''^2} \\ z' - \xi = \frac{z''^2}{x''^2+y''^2}. \end{cases} \tag{5}$$

Reasoning as in the previous section, $\mathbf{x}'$ and $\mathbf{x}''$ can be interpreted as projective rays going through two distinct origins $\mathbf{O}'$ and $\mathbf{O}''$. From equation 5 follows that the two pencils of rays intersect on a paraboloid with vertex in $\mathbf{O}''$. The equation of this paraboloid in the coordinate system attached to the origin $\mathbf{O}'$ is

$$x'^2 + y'^2 - (z' - \xi) = 0$$

.

The scheme of Fig. 3 is the equivalent to Fig 2 for the situation of lens with radial distortion. It shows an intuitive 'concrete' model for the non-linear transformation $\eth$ (Tab. 1) based on the paraboloid derived above. Since in this case the ξ parameter is always negative [8], the effective projection center $\mathbf{O}'$ lies inside the parabolic surface. The projective ray $\mathbf{x}'$ goes through the viewpoint $\mathbf{O}'$ and intersects the paraboloid at point $\mathbf{X_m}$. By joining $\mathbf{X_m}$ with the vertex $\mathbf{O}''$ we obtain the projective ray associated with the distorted image point $\mathbf{x}''$. This model is in accordance the fact that the effects of radial distortion are more noticeable in the image periphery than in the image center. Notice that the paraboloid of reference is a quadratic surface in $\wp^\mathbf{3}$ which is tangent to the plane at infinity on point $(X', Y', Z', W')^t = (0, 0, 1, 0)^t$. If the angle between the projective ray $\mathbf{x}'$ and the Z' axis is small, then the intersection point $\mathbf{X_m}$ is close to infinity. In this case the rays associated with $\mathbf{x}'$ and $\mathbf{x}''$ are almost coincident and the effect of radial distortion can be neglected..

Consider a line in the space that, according to the conventional pin-hole model, is projected into a line $\mathbf{n}' = (n'_x, n'_y, n'_z)^t$ in the projective plane. Points $\mathbf{x}'$, lying on line $\mathbf{n}'$, are transformed into image points $\mathbf{x}''$ by the non-linear function $\eth$. Since $\mathbf{n}'^t\mathbf{x}' = 0$ and $\mathbf{x}' = \eth^{-1}(\mathbf{x}'')$, then $\mathbf{n}'^t\eth^{-1}(\mathbf{x}'') = 0$. After some algebraic manipulation the previous equality can be written in the form $\mathbf{x}''^t\mathbf{\Omega}''\mathbf{x}'' = 0$ with $\mathbf{\Omega}''$ given by equation 6. In a similar way to what happens for the central catadioptric systems, the non-linear mapping $\eth$ transforms lines $\mathbf{n}'$ into a conic sections $\mathbf{\Omega}''$ (see Fig. 3).

$$\mathbf{\Omega}'' = \begin{bmatrix} \xi n'_z & 0 & \frac{n'_x}{2} \\ 0 & \xi n'_z & \frac{n'_y}{2} \\ \frac{n'_x}{2} & \frac{n'_y}{2} & n'_z \end{bmatrix} \tag{6}$$

3. Embedding $\wp^2$ into $\wp^5$ Using Veronese Maps

Perspective projection can be formulated as a transformation of $\Re^3$ into $\Re^2$. Points $\mathbf{X} = (X, Y, Z)^t$ are mapped into points $\mathbf{x} = (x, y)^t$ by a non-linear function $f(\mathbf{X}) = (X/Z, Y/Z)^t$. A standard technique used in algebra to render a nonlinear problem into a linear one is to find an embedding that lifts the problem into a higher dimensional space. For conventional cameras, the additional homogeneous coordinate linearizes the mapping function and simplifies most of the mathematic relations. In the previous section we established a unifying model that includes central catadioptric sensors and lens with radial distortion. Unfortunately the use of an additional homogeneous coordinate does no longer suffice to cope with the non-linearities in the image formation.

In this paper, we propose the embedding of the projective plane into a higher dimensional space in order to study the geometry of general single viewpoint images in a unified framework. This idea has already been explored by other authors to solve several computer vision problems. Higher-dimensional projection matrices are proposed in [18] for the representation of various applications where the world is no longer rigid. In [10], lifted coordinates are used to obtain a fundamental matrix between paracatadioptric views. Sturm generalizes this framework to analyze the relations between multiple views of a static scene where the views are taken by any mixture of paracatadioptric, perspective or affine cameras [14].

The present section discusses the embedding of the projective plane $\wp^2$ in $\wp^5$ (equation 7) using Veronese mapping [11, 12]. This polynomial embedding preserves homogeneity and is suitable to represent quadratic relations between image points [7, 16]. Moreover there is a natural duality between lifted points $\widetilde{\mathbf{x}}$ and conics which is advantageous when dealing with catadioptric projection of lines. It is also shown that projective transformations in $\wp^2$ can be transposed to $\wp^5$ in a straightforward manner.

$$\mathbf{x} \in \wp^2 \longrightarrow \widetilde{\mathbf{x}} = (x_0, x_1, x_2, x_3, x_4, x_5)^t \in \wp^5 \tag{7}$$

3.1 Lifting Point Coordinates

Consider an operator $\mathbf{\Gamma}$ which transforms two 3×1 vectors $\mathbf{x}$, $\bar{\mathbf{x}}$ into a 6×1 vector as shown in equation 8

$$\mathbf{\Gamma}(\mathbf{x}, \bar{\mathbf{x}}) = (x\bar{x}, \frac{x\bar{y} + y\bar{x}}{2}, y\bar{y}, \frac{x\bar{z} + z\bar{x}}{2}, \frac{y\bar{z} + z\bar{y}}{2}, z\bar{z})^t \tag{8}$$

The operator $\mathbf{\Gamma}$ can be used to map pairs of points in the projective plane $\wp^{\mathbf{2}}$, with homogeneous coordinates $\mathbf{x}$ and $\bar{\mathbf{x}}$, into points in the 5D projective space $\wp^{\mathbf{5}}$. To each pair of points $\mathbf{x}$, $\bar{\mathbf{x}}$ corresponds one, and only one, point $\widetilde{\mathbf{x}} = \mathbf{\Gamma}(\mathbf{x}, \bar{\mathbf{x}})$ which lies on a primal $\mathbf{S}$ called the cubic symmetroid [11]. The cubic symmetroid $\mathbf{S}$ is a non-linear subset of $\wp^{\mathbf{5}}$ defined by the following equation

$$x_0x_2x_5 + 2x_1x_3x_4 - x_0x_4^2 - x_2x_3^2 - x_5x_1^2 = 0, \forall_{\widetilde{\mathbf{x}} \in \mathbf{S}} \tag{9}$$

By making $\bar{\mathbf{x}} = \mathbf{x}$ the operator $\mathbf{\Gamma}$ can be used to map a single point in $\wp^{\mathbf{2}}$ into a point in $\wp^{\mathbf{5}}$. In this case the lifting function becomes

$$\mathbf{x} \longrightarrow \widetilde{\mathbf{x}} = \mathbf{\Gamma}(\mathbf{x}, \mathbf{x}) = (x^2, xy, y^2, xz, yz, z^2)^t. \tag{10}$$

To each point $\mathbf{x}$ in the projective plane corresponds one, and only one, point $\widetilde{\mathbf{x}}$ lying on a quadratic surface $\mathbf{V}$ in $\wp^{\mathbf{5}}$. This surface, defined by the triplet of equations 11, is called the Veronese surface and is a sub-set of the cubic symmetroid $\mathbf{S}$ [11, 12]. The mapping of equation 10 is the second order Veronese mapping that will be used to embed the projective plane $\wp^{\mathbf{2}}$ into the 5D projective space.

$$x_1^2 - x_0x_2 = 0 \ \wedge \ x_3^2 - x_0x_5 = 0 \ \wedge \ x_4^2 - x_2x_5 = 0, \forall_{\widetilde{\mathbf{x}} \in \mathbf{V}}. \tag{11}$$

3.2 Lifting Lines and Conics

A conic curve in the projective plane $\wp^{\mathbf{2}}$ is usually represented by a 3×3 symmetric matrix $\mathbf{\Omega}$. Point $\mathbf{x}$ lies on the conic if, and only if, equation $\mathbf{x}^t\mathbf{\Omega}\mathbf{x} = 0$ is satisfied. Since a 3×3 symmetric matrix has 6 parameters, the conic locus can also be represented by a 6×1 homogeneous vector $\widetilde{\boldsymbol{\omega}}$ (equation 12). Vector $\widetilde{\boldsymbol{\omega}}$ is the representation in lifted coordinates of the planar conic $\mathbf{\Omega}$

$$\mathbf{\Omega} = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \longrightarrow \widetilde{\boldsymbol{\omega}} = (a, 2b, c, 2d, 2e, f)^t. \tag{12}$$

Point $\mathbf{x}$ lies on conic the conic locus $\mathbf{\Omega}$ if, and only if, its lifted coordinates $\widetilde{\mathbf{x}}$ are orthogonal to vector $\widetilde{\boldsymbol{\omega}}$ and $\widetilde{\boldsymbol{\omega}}^t.\widetilde{\mathbf{x}} = 0$. Moreover, if points $\mathbf{x}$ and $\bar{\mathbf{x}}$ are harmonic conjugates with respect to the conic then $\mathbf{x}^t\mathbf{\Omega}\bar{\mathbf{x}} = 0$ and $\widetilde{\boldsymbol{\omega}}^t.\mathbf{\Gamma}(\mathbf{x}, \bar{\mathbf{x}}) = 0$. In the same way as points and lines are dual entities in $\wp^{\mathbf{2}}$, there is a duality between points and conics in the lifted space $\wp^{\mathbf{5}}$. Since the general single viewpoint image of a line is a conic (equations 3 and 6), this duality will prove to be a nice and useful property.

Conic $\mathbf{\Omega} = \mathbf{m}.\mathbf{l}^t + \mathbf{l}.\mathbf{m}^t$ is composed of two lines $\mathbf{m}$ and $\mathbf{l}$ lying on the projective plane $\wp^{\mathbf{2}}$. In this case the conic is said to be degenerate, the 3×3 symmetric matrix $\mathbf{\Omega}$ is rank 2, and equation 12 becomes

$$\mathbf{\Omega} = \mathbf{m}\mathbf{l}^t + \mathbf{l}\mathbf{m}^t \longrightarrow \widetilde{\boldsymbol{\omega}} = \underbrace{\begin{bmatrix} 1&0&0&0&0&0 \\ 0&2&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&2&0&0 \\ 0&0&0&0&2&0 \\ 0&0&0&0&0&1 \end{bmatrix}}_{\widetilde{\mathbf{D}}} .\mathbf{\Gamma}(\mathbf{m}, \mathbf{l}) \tag{13}$$

In a similar way a conic locus can be composed of a single line $\mathbf{n} = (n_x, n_y, n_z)^t$. Matrix $\mathbf{\Omega} = \mathbf{n}.\mathbf{n}^t$ has rank 1 and the result of equation 12 can be used to establish the lifted representation of a line

$$\mathbf{n} \rightarrow \widetilde{\mathbf{n}} = \widetilde{\mathbf{D}}.\mathbf{\Gamma}(\mathbf{n}, \mathbf{n}) = (n_x^2, 2n_xn_y, n_y^2, 2n_xn_z, 2n_yn_z, n_z^2)^t \tag{14}$$

Consider a point $\mathbf{x}$ in $\wp^{\mathbf{2}}$ lying on line $\mathbf{n}$ such that $\mathbf{n}^t.\mathbf{x} = 0$. Point $\mathbf{x}$ is on the line if, and only if, its lifted coordinates $\widetilde{\mathbf{n}}$ are orthogonal to the homogeneous vector $\widetilde{\mathbf{n}}$ ($\widetilde{\mathbf{n}}^{\mathbf{t}}\widetilde{\mathbf{x}} = 0$). Points and lines are dual entities in $\wp^{\mathbf{2}}$ as well as in the lifted space $\wp^{\mathbf{5}}$. By embedding the projective plane into $\wp^{\mathbf{5}}$ lines and conics are treated in a uniform manner. The duality between points and lines is preserved and extended for the case of points and conics. The space of all conics is the dual 5D projective space $\wp^{\mathbf{5*}}$, because each point $\widetilde{\boldsymbol{\omega}}$ corresponds to a conic curve $\mathbf{\Omega}$ in the original 2D plane. The set of all lines $\mathbf{n}$ is mapped into a non-linear subset $\mathbf{V}^*$ of $\wp^{\mathbf{5*}}$, which is the projective transformation of the Veronese surface $\mathbf{V}$ by $\widetilde{\mathbf{D}}$ (equation 14).

3.3 Lifting Conic Envelopes

Each point conic $\mathbf{\Omega}$ has dual conic $\mathbf{\Omega^*}$ associated with it [11]. The line conic $\mathbf{\Omega^*}$ is usually represented by a 3×3 symmetric matrix and a generic line $\mathbf{n}$ belongs to the conic envelope whenever satisfies $\mathbf{n}^t\mathbf{\Omega^*}\mathbf{n} = 0$. The conic envelope can also be represented by a 6×1 homogeneous vector $\widetilde{\boldsymbol{\omega}}^*$ like the one provided in equation 15. In this case line $\mathbf{n}$ lies on $\mathbf{\Omega^*}$ if, and only if, the corresponding lifted vector $\widetilde{\mathbf{n}}$ (equation 14) is orthogonal to $\widetilde{\boldsymbol{\omega}}^*$.

$$\mathbf{\Omega^*} = \begin{bmatrix} a^* & b^* & d^* \\ b^* & c^* & e^* \\ d^* & e^* & f^* \end{bmatrix} \longrightarrow \widetilde{\boldsymbol{\omega}}^* = (a^*, b^*, c^*, d^*, e^*, f^*)^t. \tag{15}$$

If matrix $\mathbf{\Omega^*}$ is rank deficient then the conic envelope is said to be degenerate. There are two possible cases of degeneracy: when the line conic is composed by two pencils of lines going through a pair of points $\mathbf{x}$ and $\bar{\mathbf{x}}$, and when the conic envelope is composed by a single pencil of lines. In the former case $\mathbf{\Omega}^* = \mathbf{x}\bar{\mathbf{x}}^t + \bar{\mathbf{x}}\mathbf{x}^t$ and the lifted representation becomes

$$\mathbf{\Omega}^* = \mathbf{x}\bar{\mathbf{x}}^t + \bar{\mathbf{x}}\mathbf{x}^t \longrightarrow \widetilde{\omega}^* = \mathbf{\Gamma}(\mathbf{x}, \bar{\mathbf{x}}) \tag{16}$$

If the line conics is a single pencil going through point $\mathbf{x}$ then $\mathbf{\Omega}^* = \mathbf{x}\mathbf{x}^t$ and

$$\mathbf{\Omega}^* = \mathbf{x}\mathbf{x}^t \longrightarrow \widetilde{\omega}^* = \mathbf{\Gamma}(\mathbf{x}, \mathbf{x}) \tag{17}$$

3.4 Lifting Linear Transformations

On the previous sections we discussed the representation of points, lines, conics and conic envelopes in the 5D projective space $\wp^{\mathbf{5}}$. However a geometry is defined not only by a set of objects but also by the group of transformations acting on them [5]. This section shows how a linear transformation on the original space $\wp^{\mathbf{2}}$ can be coherently transferred to the lifted space $\wp^{\mathbf{5}}$.

Consider a linear transformation, represented by a 3×3 matrix $\mathbf{H}$, which maps any two points $\mathbf{x}$, $\bar{\mathbf{x}}$ into points $\mathbf{H}\mathbf{x}$, $\mathbf{H}\bar{\mathbf{x}}$. Both pairs of points can be lifted to $\wp^{\mathbf{5}}$ using the operator $\mathbf{\Gamma}$ of equation 8. We wish to obtain a new operator $\mathbf{\Lambda}$ that has the following characteristic

$$\mathbf{\Gamma}(\mathbf{H}\mathbf{x}, \mathbf{H}\bar{\mathbf{x}}) = \mathbf{\Lambda}(\mathbf{H}).\mathbf{\Gamma}(\mathbf{x}, \bar{\mathbf{x}}) \tag{18}$$

The desired result can be derived by developing equation 18 and performing some algebraic manipulation. The operator $\mathbf{\Lambda}$, transforming a 3×3 matrix $\mathbf{H}$ into a 6×6 matrix $\widetilde{\mathbf{H}}$, is provided in equation 19 with $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ denoting the columns of the original matrix $\mathbf{H}$.

$$\mathbf{\Lambda}(\underbrace{\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3}\end{bmatrix}}_{\mathbf{H}}) = \underbrace{\begin{bmatrix}\mathbf{\Gamma}(\mathbf{v_1},\mathbf{v_1})^t \\ \mathbf{\Gamma}(\mathbf{v_1},\mathbf{v_2})^t \\ \mathbf{\Gamma}(\mathbf{v_2},\mathbf{v_2})^t \\ \mathbf{\Gamma}(\mathbf{v_1},\mathbf{v_3})^t \\ \mathbf{\Gamma}(\mathbf{v_2},\mathbf{v_3})^t \\ \mathbf{\Gamma}(\mathbf{v_3},\mathbf{v_3})^t\end{bmatrix} \widetilde{\mathbf{D}}}_{\widetilde{\mathbf{H}}} \tag{19}$$

It can be proved that $\mathbf{\Lambda}$, not only satisfies the relation stated on equation 18, but also has the following properties

$$\begin{array}{rcl}\mathbf{\Lambda}(\mathbf{H}^{-1}) & = & \mathbf{\Lambda}(\mathbf{H})^{-1} \\ \mathbf{\Lambda}(\mathbf{H}.\mathbf{B}) & = & \mathbf{\Lambda}(\mathbf{H}).\mathbf{\Lambda}(\mathbf{B}) \\ \mathbf{\Lambda}(\mathbf{H}^t) & = & \widetilde{\mathbf{D}}^{-1}.\mathbf{\Lambda}(\mathbf{H})^t.\widetilde{\mathbf{D}} \\ \mathbf{\Lambda}(\mathbf{I}_{3\times3}) & = & \mathbf{I}_{6\times6}\end{array} \tag{20}$$

From equation 18 comes that if $\mathbf{x}$ and $\mathbf{y}$ are two points in $\wp^{\mathbf{2}}$ such that $\mathbf{y} = \mathbf{H}\mathbf{x}$ then $\widetilde{\mathbf{y}} = \mathbf{\Lambda}(\mathbf{H}).\widetilde{\mathbf{x}}$ where $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ are the lifted coordinates of the points. The operator $\mathbf{\Lambda}$ maps the linear transformation $\mathbf{H}$ in the plane into the linear transformation $\widetilde{\mathbf{H}} = \mathbf{\Lambda}(\mathbf{H})$ in $\wp^{\mathbf{5}}$. The transformation of points, conics and conic envelopes are transferred to the 5D projective space in the following manner

$$\begin{array}{lll} \mathbf{y} = \mathbf{H}\mathbf{x} & \longrightarrow & \widetilde{\mathbf{y}} = \widetilde{\mathbf{H}}\widetilde{\mathbf{x}} \\ \mathbf{\Psi} = \mathbf{H}^{-t}\mathbf{\Omega}\mathbf{H}^{-1} & \longrightarrow & \widetilde{\boldsymbol{\psi}} = \widetilde{\mathbf{H}}^{-t}\widetilde{\boldsymbol{\omega}} \\ \mathbf{\Psi}^{*} = \mathbf{H}\mathbf{\Omega}^{*}\mathbf{H}^{t} & \longrightarrow & \widetilde{\boldsymbol{\psi}}^{*} = \widetilde{\mathbf{H}}\widetilde{\boldsymbol{\omega}}^{*} \end{array} \tag{21}$$

The operator $\mathbf{\Lambda}$ can be applied to obtain a lifted representations for both collineations and correlations. A correlation $\mathbf{G}$ in $\wp^{\mathbf{2}}$ transforms a point $\mathbf{x}$ into a line $\mathbf{n} = \mathbf{G}\mathbf{x}$. From equations 10 and 14 the lifted coordinates for $\mathbf{x}$ and $\mathbf{n}$ are respectively $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{n}}$. It comes in a straightforward manner that the lifted vectors are related in $\wp^{\mathbf{5}}$ by $\widetilde{\mathbf{n}} = \widetilde{\mathbf{D}}\widetilde{\mathbf{G}}\widetilde{\mathbf{x}}$. Thus the correlation $\mathbf{G}$ in $\wp^{\mathbf{2}}$ is represented in the 5D projective space by $\widetilde{\mathbf{D}}\widetilde{\mathbf{G}}$ with $\widetilde{\mathbf{G}} = \mathbf{\Lambda}(\mathbf{G})$ and $\widetilde{\mathbf{D}}$ the diagonal matrix of equation 13.

We just proved that the set of linear transformations in $\wp^{\mathbf{2}}$ can be mapped into a subset of linear transformations in $\wp^{\mathbf{5}}$. Any transformation, represented by a singular or non-singular 3×3 matrix $\mathbf{H}$, has a correspondence in $\widetilde{\mathbf{H}} = \mathbf{\Lambda}(\mathbf{H})$. However note that there are linear transformations in $\wp^{\mathbf{5}}$ without any correspondence in the projective plane.

4. The Subset of Line Images

This section applies the established framework in order to study the properties of line projection in central catadioptric systems and cameras with radial distortion. If it is true that a line is mapped into a conic in the image, it is not true that any conic can be the projection of a line. It is shown that a conic section $\widetilde{\boldsymbol{\omega}}$ is the projection of a line if, and only if, it lies in a certain subset of $\wp^{\mathbf{5}}$ defined by the sensor type and calibration. This subset is a linear subspace for paracatadioptric cameras and cameras with radial distortion, and a quadratic surface for hyperbolic systems.

4.1 Central Catadioptric Projection of Lines

Assume that a certain line in the world is projected into a conic section $\mathbf{\Omega}''$ in the catadioptric image plane. As shown in Fig. 2 the line lies in plane $\mathbf{\Pi}$ that contains the projection center $\mathbf{O}$ and is orthogonal to $\mathbf{n} = (n_x, n_y, n_z)^t$. The catadioptric projection of the line is $\mathbf{\Omega}'' = \mathbf{H_c}^{-t}\mathbf{\Omega}'\mathbf{H_c}^{-1}$ where $\mathbf{H_c}$ is the calibration matrix. The conic $\mathbf{\Omega}'$ is provided in equation 3 and depends on the normal $\mathbf{n}$ and the shape of the mirror.

The framework derived in the previous section is now used to transpose to $\wp^{\mathbf{5}}$ the model for line projection discussed in section 2.1. Conic $\mathbf{\Omega'}$ is mapped into $\boldsymbol{\omega'}$ in the 5D projective space. As shown in equation 3 the conic depends on the normal $\mathbf{n}$ and on parameter ξ. This dependence can be represented in $\wp^{\mathbf{5}}$ by $\widetilde{\boldsymbol{\omega}}' = \widetilde{\boldsymbol{\Delta}}_{\boldsymbol{c}}\widetilde{\mathbf{n}}$ with $\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{c}}$ given by equation 22. The lifted coordinates of the final image of the line are $\widetilde{\boldsymbol{\omega}}'' = \widetilde{\mathbf{H}}_{\mathbf{c}}\widetilde{\boldsymbol{\Delta}}_{\mathbf{c}}\widetilde{\mathbf{n}}$. Hence forth, if nothing is said, the collineation $\widetilde{\mathbf{H}}_{\mathbf{c}}$ is ignored and we will work directly with $\widetilde{\boldsymbol{\omega}}' = \widetilde{\mathbf{H}}_{\mathbf{c}}^{-1}\widetilde{\boldsymbol{\omega}}''$.

$$\underbrace{\begin{bmatrix} a' \\ 2b' \\ c' \\ 2d' \\ 2e' \\ f' \end{bmatrix}}_{\widetilde{\boldsymbol{\omega}}'} = \underbrace{\begin{bmatrix} 1-\xi^2 & 0 & 0 & 0 & 0 & -\xi^2 \\ 0 & 1-\xi^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\xi^2 & 0 & 0 & -\xi^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{c}}} \underbrace{\begin{bmatrix} n_x^2 \\ 2n_xn_y \\ n_y^2 \\ 2n_xn_z \\ 2n_yn_z \\ n_z^2 \end{bmatrix}}_{\widetilde{\boldsymbol{n}}} \qquad (22)$$

Notice that the linear transformation $\widetilde{\boldsymbol{\Delta}}_{\mathbf{c}}$, derived from equation 3, does not have an equivalent transformation in the projective plane (equation 19). The catadioptric projection of a line, despite of being non-linear in $\wp^{\mathbf{2}}$, is described by a linear relation in $\wp^{\mathbf{5}}$.

As stated in section 3.2, a line $\mathbf{n}$ in the projective plane is lifted into a point $\widetilde{\mathbf{n}}$ which lies on the quadratic surface $\mathbf{V}^*$ in $\wp^{\mathbf{5}*}$. From equation 22 it follows that conic $\widetilde{\boldsymbol{\omega}}'$ is the catadioptric projection of a line if, and only if, $\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{c}}^{-1}\widetilde{\boldsymbol{\omega}}' \in \mathbf{V}^*$. Since surface $\mathbf{V}^*$ is the projective transformation of the Veronese surface $\mathbf{V}$ (equation 11) by $\widetilde{\mathbf{D}}$, then $\widetilde{\boldsymbol{\omega}}' = (a', 2b', c', 2d', 2e', f')^t$ is the projection of a line if, and only if,

$$\begin{cases} d'^2(1-\xi^2) - f'(a' + f'\xi^2) = 0 \\ e'^2(1-\xi^2) - f'(c' + f'\xi^2) = 0 \\ b'^2 - (a' + f\xi^2)(c' + f'\xi^2) = 0 \end{cases} , \forall_{\widetilde{\boldsymbol{\omega}}' \in \boldsymbol{\zeta}} \qquad (23)$$

Equation 23 defines a quadratic surface $\boldsymbol{\zeta}$ in the space of all conics. The constraints of equation 23 have been recently introduced in [19] and used as invariants for calibration purposes.

4.2 Line Projection in Paracatadioptric Cameras

Let's consider the situation of paracatadioptric cameras where $\xi = 1$. In this case point $\mathbf{O}'$ lies on the sphere surface (Fig. 2) and the re-projection from the sphere to the plane becomes a stereographic projection [9]. Equation 24 is derived by replacing ξ in equation 23. For the particular case of paracatadioptric cameras the quadratic surface $\boldsymbol{\zeta}$ degenerates to a linear subspace $\boldsymbol{\varphi}$ which the set of all line projections $\widetilde{\boldsymbol{\omega}}'$.

$$a' + f' = 0 \ \wedge \ c' + f' = 0 \ \wedge \ b'^2 = 0, \forall_{\widetilde{\boldsymbol{\omega}}' \in \boldsymbol{\varphi}} \tag{24}$$

Stating this result in a different way, the conic $\boldsymbol{\Omega}'$ is is the paracatadioptric projection of a line if, and only if, the corresponding lifted representation $\widetilde{\boldsymbol{\omega}}'$ is on the null space of matrix $\mathbf{N_p}$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{N_p}} \widetilde{\boldsymbol{\omega}}' = 0 \tag{25}$$

We have already seen that if point $\mathbf{x}'$ is on conic $\boldsymbol{\Omega}'$ then $\widetilde{\boldsymbol{\omega}}'^t \widetilde{\mathbf{x}}' = 0$. In $\wp^{\mathbf{5}}$ the lifted coordinates $\widetilde{\mathbf{x}}'$ must lie on the prime orthogonal to $\widetilde{\boldsymbol{\omega}}'$ [11]. However, not all points in this prime are lifted coordinates of points in $\wp^{\mathbf{2}}$. Section 3.1 shows that only points lying on the Veronese surface $\mathbf{V}$ have a correspondence on the projective plane. Thus, points $\mathbf{x}'$ lying on $\boldsymbol{\Omega}'$ are mapped into a subset of $\wp^{\mathbf{5}}$ defined by the intersection of the prime orthogonal to $\widetilde{\boldsymbol{\omega}}'$ with the Veronese surface $\mathbf{V}$.

Consider the set of all conic sections $\boldsymbol{\Omega}'$ corresponding to paracatadioptric line projections. If this conic set has a common point $\mathbf{x}'$ then its lifted vector $\widetilde{\mathbf{x}}'$ must be on the intersection of $\mathbf{V}$ with the hyperplane orthogonal to $\boldsymbol{\varphi}$. Points $\widetilde{\mathbf{I}}'$ and $\widetilde{\mathbf{J}}'$ are computed by intersecting the range of matrix $\mathbf{N_p}^t$ (the orthogonal hyperplane) with the Veronese surface defined in equation 11. These points are the lifted coordinates of the circular points in the projective plane where all paracatadioptric line images $\boldsymbol{\Omega}'$ intersect.

$$\left\{ \begin{array}{l} \widetilde{\mathbf{I}}' = (1, i, -1, 0, 0, 0)^t \\ \widetilde{\mathbf{J}}' = (1, -i, -1, 0, 0, 0)^t \end{array} \right. \rightarrow \left\{ \begin{array}{l} \mathbf{I}' = (1, i, 0)^t \\ \mathbf{J}' = (1, -i, 0)^t \end{array} \right. \tag{26}$$

In a similar way, if there is a pair of points $\mathbf{x}, \bar{\mathbf{x}}$ that are harmonic conjugate with respect to all conics $\boldsymbol{\Omega}'$ then, the corresponding vector $\boldsymbol{\Gamma}(\mathbf{x}, \bar{\mathbf{x}})$, must be in the intersection of $\mathbf{S}$ with the range of $\mathbf{N_p}^t$. The intersection can be determined from equations 9 and 25 defining the cubic symmetroid $\mathbf{S}$ and matrix $\mathbf{N_p}$. The result is presented in equation 27 where λ is a free scalar.

$$\begin{cases} \widetilde{\mathbf{P}'\mathbf{Q}'} & = (-\lambda, 1, \lambda, 0, 0, 0)^t \rightarrow \begin{cases} \mathbf{P}' = (1 + \sqrt{1+\lambda^2}, \lambda, 0)^t \\ \mathbf{Q}' = (1 - \sqrt{1+\lambda^2}, \lambda, 0)^t \end{cases} \\ \\ \widetilde{\mathbf{R}'\mathbf{T}'} & = (1, \lambda, \lambda^2, 0, 0, 1+\lambda^2)^t \rightarrow \begin{cases} \mathbf{R}' = (1, \lambda, -i\sqrt{1+\lambda^2})^t \\ \mathbf{T}' = (1, \lambda, i\sqrt{1+\lambda^2})^t \end{cases} \end{cases} \quad (27)$$

According to equation 26, any paracatadioptric projection of a line must go through the circular points. This is not surprising, since the stereographic projection of a great circle is always a circle (see 2). However, not all circles correspond to the projection of lines. While points $\mathbf{P}', \mathbf{Q}'$ are harmonic conjugate with respect to a all circles, the same does not happen with the pair $\mathbf{R}', \mathbf{T}'$. Thus, a conic $\mathbf{\Omega}'$ is the paracatadioptric image of a line if, and only if, it goes through the circular points and satisfies $\mathbf{R}'^t\mathbf{\Omega}'\mathbf{T}' = 0$. This result has been used in [3, 2] in order to constrain the search space and accurately estimate line projections in the paracatadioptric image plane.

4.3 Line Projection in Cameras with Radial Distortion

We have already shown that for catadioptric cameras the model for line projection becomes linear when the projective plane is embedded in $\wp^{\mathbf{5}}$. A similar derivation can be applied to dioptric cameras with radial distortion. According to the conventional pin-hole model a line in the scene is mapped into a line $\mathbf{n}'$ in the image plane. However, and as discussed on section 2.2, the non-linear effect of radial distortion transforms $\mathbf{n}'$ into a conic curve $\mathbf{\Omega}''$. If $\widetilde{\boldsymbol{\omega}}''$ and $\widetilde{\mathbf{n}}''$ are the 5D representations of $\mathbf{\Omega}''$ and $\mathbf{n}'$ it comes from equation 6 that

$$\underbrace{\begin{bmatrix} a'' \\ 2b'' \\ c'' \\ 2d'' \\ 2e'' \\ f'' \end{bmatrix}}_{\widetilde{\boldsymbol{\omega}}''} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \xi \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\widetilde{\mathbf{\Delta}}_{\boldsymbol{r}}} \underbrace{\begin{bmatrix} n_x'^2 \\ 2n_x'n_y' \\ n_y'^2 \\ 2n_x'n_z' \\ 2n_y'n_z' \\ n_z'^2 \end{bmatrix}}_{\widetilde{\boldsymbol{n}}'} \quad (28)$$

Consider matrix $\widetilde{\mathbf{\Delta}}_{\boldsymbol{c}}$ for the paracatadioptric camera situation with $\xi = 1$. The structure of $\widetilde{\mathbf{\Delta}}_{\boldsymbol{r}}$ and $\widetilde{\mathbf{\Delta}}_{\boldsymbol{c}}$ is quite similar. It can be proved that a conic section $\widetilde{\boldsymbol{\omega}}''$ is the distorted projection of a line if, and only if, it lies on a hyperplane ς defined as follows

$$a'' - \xi f'' = 0 \ \wedge \ c'' - \xi f'' = 0 \ \wedge \ b''^2 = 0, \forall_{\widetilde{\boldsymbol{\omega}}' \in \varsigma} \quad (29)$$

Repeating the reasoning that we did for the paracatadioptric camera, it can be shown that conic $\mathbf{\Omega''}$ is the distorted projection of a line if, and only if, it goes through the circular points of equation 26 and satisfies the condition $\mathbf{M''}^t\mathbf{\Omega''}\mathbf{N''} = 0$ with $\mathbf{M''}$ and $\mathbf{N''}$ given below

$$\widetilde{\mathbf{M''N''}} = (1, \lambda, \lambda^2, 0, 0, -\xi(1+\lambda^2))^t \rightarrow \begin{cases} \mathbf{M''} = (1, \lambda, \sqrt{\xi(1+\lambda^2)})^t \\ \mathbf{N''} = (1, \lambda, -\sqrt{\xi(1+\lambda^2)})^t \end{cases} \tag{30}$$

5. Conclusion

In this paper we studied unifying models for central projection systems and representations of projections of world points and lines. We first proved that the two step projection model through the sphere, equivalent to perspective cameras and all central catadioptric systems, can be extended to cover the division model of radial lens distortion. Having accommodated all central catadioptric as well as radial lens distortion models under one formulation, we established a representation of the resulting image planes in the five-dimensional projective space through the Veronese mapping. In this space, a collineation of the original plane corresponds to a collineation of the lifted space. Projections of lines in the world correspond to points in the lifted space lying in the general case on a quadric surface. However, in the cases of paracatadioptric and radial lens distortions, liftings of the projections of world lines lie on hyperplanes. In ongoing work, we study the epipolar geometry of central camera systems when points are expressed in this lifted space.

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